Study materials of Mathematics for class D-III (H), Paper - V on the topic " C-R Equations in polar form " presented by Dr. Md. Shamim Ahmad, Associate Professor, U.R. College Rosera dated 30.04.21

C-R Equations in Polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof. We know that $x = r\cos\theta$ and $y = r\sin\theta$ & u, v are functions of x & y.

$$\therefore z = x+iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$u+iv = f(z) = f(re^{i\theta}) \quad \text{--- (1)}$$

diff (1) partially w.r.to r we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{\partial f(re^{i\theta})}{\partial f(re^{i\theta})} \cdot \frac{\partial(re^{i\theta})}{\partial r}$$

$$= f'(re^{i\theta}) \cdot e^{i\theta} \cdot 1 = f'(re^{i\theta})e^{i\theta} \quad \text{--- (2)}$$

Again diff (1) partially w.r. to θ, we have

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot \frac{\partial(re^{i\theta})}{\partial \theta}$$

$$= f'(re^{i\theta}) \cdot r \cdot e^{i\theta} i \quad \text{--- (3)}$$

$$= ri \cdot \left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \text{ using (2)}$$

$$= ri\frac{\partial u}{\partial r} + ri^2\frac{\partial v}{\partial r}$$

$$= ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real & imaginary parts, we have

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \Rightarrow \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

and $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$ Proved.

Laplace's differential equation

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Let $\phi(x,y)$ be a function of two variables x & y which admits of second order partial derivatives & they are continuous functions of x & y.

Then equation $\boxed{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0}$ is called Laplace's differential equation

Theorem: Real & imaginary parts of an analytic function satisfy Laplace's equation.

Proof: Let the function

$f(z)=u+iv$

be analytic in some domain D. Then Cauchy-Riemann equations give $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ ——— (1)

Now, Let us assume that the second order partial derivatives of u & v exist and are continuous functions of x & y.

Then from (1), we have $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}$ & $\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x}$

Adding them, we get $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ——— (2)

Similarly $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$ ——— (3)

The equations (2) & (3) show that the function u & v satisfy the Laplace's differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Harmonic functions

Def^n: Any function which satisfies the Laplace's equation is known as a harmonic function.

Theorem: If $f(z)=u+iv$ is an analytic function, then u & v are both harmonic functions.

Proof. Let $f(z) = u+iv$ be an analytic function, then we have by C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (1)}$$

$$\& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (2)}$$

Diff (1) partially w.r.to x, we have $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ --- (3)

Diff (2) partially w.r.to y, we have $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$ --- (4)

Adding (3) & (4), we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$= 0 \quad \left[\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right]$$

Similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Therefore both u and v are harmonic functions. Such functions u, v are called conjugate harmonic function if $u+iv$ is also analytic